Stress-constrained topology optimization

Matteo Bruggi, Department of Civil and Environmental Engineering DICA, Politecnico di Milano, Milano, Italy
Structural and topology optimization

a) sizing optimization:
   the areas of the elements of a fixed truss “ground structure” are unknown
b) shape optimization:
   the parameters describing the geometry of the boundaries are unknown
c) topology optimization:
   the distribution of material is unknown

(Topology Optimization: Methods and Applications, Bendsøe and Sigmund, 2003)
Structural topology optimization: a design tool

Airbus A320 nacelle hinge: conventional layout vs. 3D-printed topology optimized design (EADS Airbus Defence and Space 2014)

Joris Laarman’s Bone chair takes its inspiration from topology optimization: “adding material where stiffness is needed and taking away material where it’s unnecessary”
Structural topology optimization: a design tool

Gigantic tree-like columns support the overhanging roof canopy of the Qatar National Convention Centre by Japanese Architect Arata Isozaki
Structural topology optimization: a design tool

A one-of-a-kind project: A conceptual design for the Zendai competition (China) created with topology optimization by Prof. Glaucio Paulino's research group along with Skidmore, Owings & Merrill LLP

Illustration for the concept design of a 288 m tall high-rise in Australia, showing the engineering and architecture expressed together at Skidmore, Owings & Merrill LLP (Beghini, Beghini, Katz, Baker and Paulino, 2014)
Motivations

- Most of the conventional formulations for topology optimization of two and three-dimensional elastic bodies address objective functions and constraints related to the structural stiffness, while neglecting the material strength:
  - *minimum compliance with volume constraint* (Bendsøe and Kikuchi, 88)

- Methods implementing stress enforcements deal with materials having a symmetric behavior in tension and compression, adopting the Von Mises criterion as equivalent stress to be constrained:
  - *minimum weight with Von Mises stress constraints* (Duysinx and Bendsøe, 98)

- This talk addresses the adoption of a *minimum weight* formulation implementing both *compliance and Drucker-Prager stress constraints* for the design of two-dimensional structures with equal/unequal behavior in tension and compression. (B and Duysinx, 2012)
Governing equations:
- “penalized” elasticity problem and structural compliance
- D-P failure criterion for materials with unequal behavior in t/c

The “singularity problem” in stress-constrained optimization:
- truss design / topology optimization
- mathematical relaxations

Problem formulations:
- conventional volume-constrained minimum compliance formulation
- novel energy-based stress-constrained formulation

Applications:
- design of stiff structures with strength constraints (unilateral material/support)
- design of stiff structures with simplified fatigue constraints
- optimal fiber-reinforcement for structural strengthening
Governing equations: “penalized” elasticity problem

Given a domain with assigned loads and boundary conditions, find the distribution of isotropic linear elastic material that minimizes an assigned scalar function for a fixed set of constraints

\[ 0 < \rho(\chi) \leq 1 \]

Function representing the material density, i.e. the minimization unknown

SIMP: Solid Isotropic Microstructure with Penalty (Rozvany et al, 92)

i.e. the material model to interpolate the constitutive tensor depending on \( r \)

Elasticity tensor of the given isotropic material (“full material”) \( p > 1 \) to penalize intermediate densities and achieve a pure 0-1 design (Bendsøe and Sigmund, 99)

The elastic problem depends on \( \rho(\chi) \): find \( u \in H^1 \) such that \( u \big|_{\Gamma_u} = u_0 \) and

\[
\int_{\Omega} \rho^p C_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, d\Omega = \int_{\Gamma_t} t_0 \cdot v \, d\Gamma \quad \forall v \in H^1
\]
A classical scheme for the discretization of the above problem adopts the same mesh of four nodes elements for a two-field interpolation with:

- A piecewise constant density discretization (with unknowns $x$)
- A bilinear displacement approximation (with unknowns $U$)

$$K(x) \ U = F$$

**topology optimization → sizing optimization**

- **Compliance:**
  - work of the external loads at equilibrium (Clapeyron th.)
  - measure of the structural stiffness

$$C = \int_{\Omega} \rho_h^p C^0_{ijkl} \varepsilon_{ij}(u_h)\varepsilon_{kl}(u_h) \ d\Omega = \ U^T K U = \sum_{e=1}^{N} x^p_e U^e_e K^0_e U_e,$$

- $U_e$: element displacements vector
- $K_e$: element stiffness matrix for virgin material
Governing equations:
failure criterion for materials with unequal behavior in t/c

- Failure criterion based on the Drucker-Prager equivalent stress

\[ \sigma^{eq} = \frac{s + 1}{2s} \sqrt{3J_{2D}} + \frac{s - 1}{2s}J_1 \leq \sigma_L \]

\[ s = \frac{\sigma_{Lc}}{\sigma_{Lt}} \quad (\text{Drucker and Prager, 52}) \]

- \( J_{2D} \): second deviatoric stress invariant
- \( J_1 \): first stress invariant
- \( s \): uniaxial asymmetry ratio of the strength of the material in compression to tension

computed on: \( \sigma_{ij} = \rho_h^P C_{ijkl}^0 \varepsilon_{kl}(u_h) \)

+ It handles unequal material behavior in t/c, even in case of extreme values of \( s \)
+ It is a smooth criterion
+ It reduces to Von Mises eq. stress for \( s=1 \)

Strength domain in the plane of the principal stresses for “standard” choices of the parameters (see e.g. Luo and Kang 2012, Amstutz et al. 2012)
Governing equations: failure criterion for materials with unequal behavior in t/c

- The “macroscopic” D-P equivalent stress for the e-th finite element reads:

\[ \sigma_{eq}^e = \sigma_p^e \left( \frac{s+1}{2s} \sqrt{U^T_e M^0_e U_e} + \frac{s-1}{2s} H^0_e U_e \right) = \sigma^e \sigma_{eq}^e \]

M\(_e\) : “Von Mises stress matrix”
H\(_e\) : “Hydrostatic stress matrix”

An appropriate failure criterion for the porous SIMP material should be defined on the so-called apparent “local” stress, i.e. \( \langle \sigma_{ij} \rangle = \sigma_{ij} / \sigma^q_e \) (Duysinx and Bendsøe, 98)

Element-wise stress constraints may be re-written as:

\[ \frac{\langle \sigma_{eq}^e \rangle}{\sigma_{Lt}} = \frac{\sigma_{eq}^e}{\sigma_{Lt}} \leq 1 \]

- If \( p=q \), physical consistence of the SIMP material is preserved but the singularity problem arises (Kirsch, 90):

\[ \lim_{\sigma_{eq}^e \to 0} \langle \sigma_{eq}^e \rangle = \sigma^e (p-q) / \sigma_{eq}^e \neq 0 \]

Quasi-zero densities have non-zero stresses
The singularity problem: topology optimization

- A classical topology optimization problem: find the distribution of material that minimizes weight under stress constraints

\[ \frac{\langle \sigma_{eq} \rangle}{\sigma_{Lt}} \leq 1 \]

Two-bar truss (Duysinx and Bendsøe, 98)

- Gradient-based minimizers are not able to remove material with low or zero density: undesired local minima with extended grey areas arise

- Reasons for the singularity problem should be investigated moving to sizing optimization
A classical sizing problem: find the truss areas that minimize the weight subject to local stress constraints due to multiple load cases.

Three-bar truss (Cheng & Jiang, 92)

- $x_1 (= x_3)$ and $x_2$ are the truss density unknowns (topology) instead of truss sectional areas (sizing).
- Stress constraints are written in analytical form, with $\sigma_{ij}$ stress in the i-th bar for the j-th load case.
- The minimum weight solution is B, with $x_2 = 0$.

Feasible domain: two-dimensional region A + degenerate region BC.
The singularity problem: relaxation of stress constraints

- Degenerate appendixes cannot be handled by gradient-based algorithms that get stuck in local optima (with grey regions) instead of the expected global one

\[ \frac{\sigma_{eq}}{\sigma_{Lt}} \leq 1 \]

A mathematical relaxation of the constraint inequalities \( \frac{\sigma_{eq}}{\sigma_{Lt}} \leq 1 \) is needed

- Classical approach, ‘\( \varepsilon \)-relaxation’: (Cheng e Guo, 97)

\[ \frac{\overline{\sigma}_{eq}(s)}{\sigma_{Lt}} \leq 1 + \frac{\varepsilon}{x_e}, \quad \text{with} \quad \varepsilon^2 = x_{min} < x_e \]

- Alternatively, ‘\( qp \)-approach’: (B, 2008)

\[ x_e^{(p-q)} \frac{\overline{\sigma}_{eq}(s)}{\sigma_{Lt}} \leq 1, \quad \text{with} \quad q < p \]

- works as an improved ‘\( \varepsilon \)-relaxation’ depending on \( x_e \): \( \varepsilon(x_e) = x_e \left( x_e^{(q-p)} - 1 \right) \)
The singularity problem: relaxation of stress constraints for truss design

The $\varepsilon$-relaxation manipulates constraints only in the vicinity of the singularity, whereas the $qp$-approach involves wider regions, thus improving convergence to the global optimum.
The singularity problem: relaxation of stress constraints for topology optimization

\[ \frac{\sigma_{eq}^e(s)}{\sigma_L t} \leq 1 + \frac{\varepsilon}{x_e} \]

\[ \frac{\langle \sigma_{eq}^e \rangle}{\sigma_L t} \leq 1 \]

\[ \frac{\sigma_{eq}^e(s)}{\sigma_L t} \leq x_e^{(q-p)} \]

The \( qp \)-approach introduces zero bias at full density while providing large relaxation at low densities.
The singularity problem: relaxation of stress constraints for topology optimization

\[ \frac{\sigma_{e}^{eq}(s)}{\sigma_{Lt}} \leq 1 + \frac{\varepsilon}{x_{e}} \]

The solution through the \( \varepsilon \)-relaxation \( \text{qp-approach} \) is nearly independent on the relaxing parameter, less iterations are needed to achieve convergence for a larger relaxation.

\[ \frac{\langle \sigma_{e}^{eq} \rangle}{\sigma_{Lt}} \leq 1 \]

\[ \frac{\sigma_{e}^{eq}(s)}{\sigma_{Lt}} \leq x_{e}^{(q-p)} \]

The solution through the \( qp \)-approach is nearly independent on the relaxing parameter, less iterations are needed to achieve convergence for a larger relaxation.
Problem formulation: classical formulation for maximum stiffness

**MCW Minimum Compliance with Weight constraint** (Bendsøe and Kikuchi, 88)

Given a domain with assigned loads and boundary conditions, find the distribution of a prescribed amount of linear elastic material that minimizes the compliance

\[
\begin{align*}
\min_{x_{\text{min}} \leq x \leq 1} & \quad C \\
\text{s.t.} & \quad K(x) U = F, \\
& \quad W / W_0 \leq V_f
\end{align*}
\]

- Structural compliance
- Governing eqns. for the elastic problem
- Global constraint on the amount of available material

- For low \(V_f\), Strut-and-Tie models arise to achieve stiff structure or find optimal load paths

UNI EN 1992-1-1 (Eurocode 2 for reinforced concrete structures) 5.6.4: STMs through energy-based criteria
Problem formulation: stress-constrained formulations

**MWCS** Minimum Weight with Compliance and Stress constraints

\[
\begin{align*}
\min_{x_{\text{min}} \leq x_e \leq 1} & \quad W = \sum_{N} x_e V_e \\
\text{s.t.} & \quad K(x) U = F, \\
& \quad C / (\alpha_C C_0) \leq 1, \\
& \quad x_e^{(p-q)} \frac{\sigma_e^{eq}(s)}{\sigma_{Lt}} \leq 1,
\end{align*}
\]

Weight

Governing eqns. for the elastic problem

Global constraint on the ratio of the design compliance (C) to the full domain compliance (C_0), with \(\alpha_C C_0 = C_L\)

Set of relaxed element-wise local constraints based on the Drucker-Prager equivalent stress

**MWC** Minimum Weight with Compliance const.

**MWS** Minimum Weight with Stress consts.
Problem formulation: stress-constrained formulations

**MWCS** Minimum Weight with Compliance and Stress constraints

\[
\begin{align*}
\min_{x_{\text{min}} \leq x_e \leq 1} & \quad W = \sum_{c} x_e V_e \\
\text{s.t.} & \quad K(x) U = F, \\
& \quad C / C_L \leq 1, \\
& \quad x^{(n-q)} \frac{\bar{\sigma}_e^q(s)}{\sigma_{Lt}} \leq 1,
\end{align*}
\]

A design suitable for both serviceability and collapse limit state is achieved within the same formulation.

- **MWCS is an “extended” energy-based problem:**
  - the global compliance constraint drives the formulation, whereas local constraints steer the minimizer towards a feasible solution in terms of strength
  - selection strategies detect limited sets of “dangerous” stress constraints and reduce the computational burden
    (alternatively, global/aggregated constraints see e.g. Le et al 2010, Luo et al 2013, Jeong et al 2014-15)

- **MWCS is well-suited to improve MCW designs where they need** (ground constraints, singularities that interfere with a homogeneous stress distributions):
  - the problem can be simpler than a MCWS, since no limit on \( W \) is enforced
Problem formulation: numerical issues

- MMA (Svanberg 87), a method of sequential convex programming is adopted to cope with the discrete setting. The MMA sub-problems call for sensitivity information at the current iteration point that may be computed at low cost via the "adjoint method":

\[
\frac{\partial C}{\partial x_k} = -p x_k^{p-1} U_k^T K_k^0 U_k \quad \text{Compliance sensitivity (from the current displacements)}
\]

\[
\frac{\partial \langle \sigma_{eq}^e \rangle}{\partial x_k} = \delta_{ek} (p - q) x_e^{p-q-1} \sigma_{eq}^e + \frac{\partial \sigma_{eq}^e}{\partial x_k} x_e^{p-q} \quad \text{Sensitivity for the selected set of constraints (from the solution of the equilibrium equation with different r.h.s.)}
\]

\[
\frac{\partial \bar{\sigma}_{eq}}{\partial x_k} = -\bar{U}^T \frac{\partial K}{\partial x_k} U, \quad \text{where} \quad K\bar{U} = \left[ \frac{s + 1}{2s} (U^T M_0^e U) - \frac{1}{2} M_0^e U + \frac{s - 1}{2s} H_0^e \right]
\]
Problem formulation: numerical issues

- Checkerboard patterns and mesh dependence (Bruns and Tortorelli, 2001)

Checkboarded layouts are optimal solutions from a mathematical point of view, but they are not feasible from a physical point of view (depends on the adopted FEM and density discretizations).

Mesh dependence: different solutions are achieved for different meshes.
Filtered unknowns on a fixed radius $r$ are successfully used instead of the element-wise densities $x_e$ to avoid numerical instabilities, i.e. mesh dependence and checkerboard patterns (Bruns and Tortorelli, 2001)

$$\tilde{x}_e = \frac{1}{\sum_N H_{el}} \sum_N H_{el} x_l,$$

$$H_{el} = \sum_N \max(0, r_{min} - \text{dist}(e, l))$$

- $e$-th filtered unknown
- $l$-th unfiltered unknown
- distance between elements

Effect on the minimum thickness of any member of the optimal design due to increasing values of the filtering radius $r$
Numerical simulations

Example 1
The L-shaped lamina
\[ E = 1, \ \nu = 0.3 \]
Filtering radius \( r = 1/16 \)

Example 2
The two-bar truss
\[ E = 1, \ \nu = 0.3 \]
Filtering radius \( r = 1/20 \)

Example 3
The MBB-beam
\[ E = 1, \ \nu = 0.3 \]
Filtering radius \( r = 1/20 \)
Numerical simulations: the L-shaped lamina $\sigma_{Lc} = \sigma_{Lt}$ (s=1)

(a) MWC (Compliance constraint with $\alpha_c = 2.0$)

$\sigma_{\text{MAX}vm} = 9.50$

Highly stressed regions are found!

(b) MWS (Stress constraints with $\sigma_{Lc} = \sigma_{Lt} = 4.50$)

$\alpha_c = 3.7$

The stiffness is low!

(c) MWCS (Both compliance and stress constraints)

Good for both stiffness and strength
In the MWCS, a limited number of constraints is active to remove highly stressed regions. The global constraint drives the optimization, while local constraints steer the solution towards a feasible design.
Numerical simulations: the L-shaped lamina $\sigma_{Lc} = \sigma_{Lt}$ (s=1)

(a) MWC
(Compliance const. with $\alpha_c = 2.0$)
($\sigma_{\text{MAXvm}} = 14$)

(b) MWCS
(Compliance const. with $\alpha_c = 2.0$,
Stress consts. with $\sigma_{Lc} = \sigma_{Lt} = 7$)

The MWCS formulation is very efficient in removing stress peaks even for very fine meshes

16384 finite elements / density unknowns

Active constraints

\[ \text{Iteration} \]

\[ 0 \rightarrow 20 \rightarrow 40 \rightarrow 60 \rightarrow 80 \rightarrow 100 \rightarrow 120 \rightarrow 140 \]
Numerical simulations:
the two-bar truss $\sigma_{L_C} = 1/3 \sigma_{L_t}$ (s=1/3)

(a) MWC
(Compliance constraint with $a_C = 2.5$)
with $\sim 45^\circ/45^\circ$ bars
($\sigma_{\text{MAX}_t} = \sigma_{\text{MAX}_c} = 1.15$)

(b) MWS
(Stress constraints with $\sigma_{L_c} = 1$, $\sigma_{L_t} = 3$)
with $\sim 60^\circ/30^\circ$ bars
($a_C = 3.7$)

(c) MWCS
(Both compliance and stress constraints)
with $\sim 60^\circ/45^\circ$ bars

The MWS 2D result is in full agreement with that of truss optimization for unequal stress limit in tension and compression, see the equilibrium of the loaded node (Rozvany, 96)
Numerical simulations: the MBB-beam $\sigma_{Lc} = 1/3 \sigma_{Lt}$ (s=1/3)

(a) MWC compl const. $\alpha_c = 2.5$ ($\sigma_{MAXc} = 7.5$) Weight=0.360

(b) MWCS compl const. $\alpha_c = 2.5$ - Stress consts. $\sigma_{Lc} = 5.5$, $\sigma_{Lt} = 16.5$ Weight=0.365

(c) MWCS compl const. $\alpha_c = 2.5$ - Stress consts. $\sigma_{Lc} = 5.0$, $\sigma_{Lt} = 15.0$ Weight=0.372

- Stress-constrained optimal designs may be significantly different with respect to conventional pure compliance-based designs, depending on the features of the material
- A reasonable increase in terms of weight is paid in the MWCS formulations with respect to MWC
Unilateral materials

- A no-tension material calls for non-positive principal stresses:
  \[ \sigma_{ii} \leq 0, \quad \sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ij} \geq 0, \]

- An “extreme” non-symmetrical behavior in tension and compression inspires the use of “extreme” parameters within a non-symmetrical strength criterion, such as the Drucker-Prager stress-criterion

\[
\sigma^{eq} = \frac{s + 1}{2s} \sqrt{3J_{2D}} + \left[ \frac{s - 1}{2s} J_{1} \right] \leq \sigma_{Lt}
\]

\[
\begin{align*}
    s &= \frac{\sigma_{Lc}}{\sigma_{Lt}} = 100 \\
    \sigma_{Lt} &\to 0
\end{align*}
\]
Numerical simulations: no-tension material

Optimal design and relevant maps of the stress first invariant (pressure) for a benchmark example (Cai et al, 2010)

Material with symmetric behavior in tension and compression

No-tension material

Material with symmetric behavior in tension and compression – upper hinges suppressed
A limited set of constraints is active in the first iterations to steer the solution of the energy-driven optimization towards feasible designs with respect to the material behavior → low computational burden

Reasonable choices of the stress limit in tension have minor effects on CPU time without affecting the achieved design.
Unilateral supports

- A no-tension support calls for non-positive reactions along the boundary, otherwise it is inactive

\[ u_i n_i = 0, \quad \sigma_{ij} n_i n_j \leq 0, \]

or

\[ u_i n_i \neq 0, \quad \sigma_{ij} n_i n_j = 0. \]

(Unilateral contact)

- An “extreme” non-symmetrical behavior of the constraints inspires the introduction of limited two-dimensional regions in the vicinity of the boundary where “extreme” parameters of the Drucker-Prager stress-criterion prevent from the arising of elements transferring undesired reactions
Numerical simulations: no-tension support

Optimal design and relevant maps of the stress first invariant (pressure) for a variation of a benchmark example (Strömberg, 2010)

Supports with symmetric behavior in tension and compression

Unilateral (no-tension) supports
Numerical simulations: no-tension support

- A very limited set of constraints is active in the first iterations to steer the solution of the energy-driven optimization towards feasible layouts with respect to the material behavior → low computational burden
A minimum weight formulation implementing both compliance and relaxed Drucker-Prager stress constraints has been used for the optimal design of structures made with equal/unequal behaviour in tension and compression.

Numerical investigations show that:

- the algorithm is able to solve successfully benchmark examples, with no grey regions or other numerical instabilities

- the formulation provides stiff layouts that are feasible with respect to the material strength, improving compliance-based results without the full cost of a stress-constrained solution

- stress-constrained optimal designs may be significantly different with respect to conventional pure compliance-based designs, especially for the DP criterion, and can handle unilateral materials/supports
Recent developments and ongoing research

- Numerical issues: need for increased accuracy in the analysis of the stress field
  need for stable discretizations to handle incompressible materials

Adaptive Finite Element Method to increase the accuracy in the evaluation of strain and stress (B and Verani, 2011) with MOX

Virtual Element Method to avoid grid-dependent layouts as sub-optimal solutions (Antonietti, B, Scacchi, Verani, 2016) with MOX and Unimi

Truly-Mixed Finite Element Method to formulate an efficient fully stable problem in terms of stresses on regular grids (B, 2016)
Recent developments and ongoing research

- Novel applications: design of stiff structures with simplified fatigue constraints
  with University of Liège (Maxime Collet visiting PhD student at Dica in 2015 and 2016)

\[
\begin{align*}
\sigma_{max} & \quad \sigma_{min} \\
\sigma_{a} & \quad \sigma_{m}
\end{align*}
\]

Alternate and mean part of the stress

\[
\begin{align*}
\sigma_{a}^e & = \frac{1}{2} \left( \sigma_{max} \right) \\
\sigma_{m} & = \frac{1}{2} \left( \sigma_{max} + \sigma_{min} \right)
\end{align*}
\]

Goodman criterion

\[
\begin{align*}
3J_{2D,e}(\sigma_{ij}) & = x_{e}^{2}U_{e}^{T}M_{e}^{0}U_{e} \\
J_{1,e}(\sigma_{ij}) & = x_{e}^{2}H_{e}^{0}U_{e}
\end{align*}
\]

\[
\begin{align*}
\sigma_{a}^{eq} & = x_{e}^{p}c_{a}\sqrt{U_{e}^{T}M_{e}^{0}U_{e}} = x_{e}^{p}\sigma_{a,e}^{eq} \\
\sigma_{m}^{eq} & = x_{e}^{p}c_{m}H_{e}^{0}U_{e} = x_{e}^{p}\sigma_{m,e}^{eq}
\end{align*}
\]

Sines formula

M. Bruggi

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Recent developments and ongoing research

- Novel applications: optimal fiber-reinforcement for structural strengthening

Formulation of an optimization method to distribute and orientate an available amount of unidirectional fiber-reinforcement (FRP) such that in-plane stresses are feasible with a Tsai-Wu strength criterion governing the underlying anisotropic layer.

Plain/reinforced concrete and brickwork can be dealt with.
Thank you for the attention!